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THE BUCKLING OF ORTHOTROPIC RECTANGULAR PLATES, INCLUDING THE EFFECT OF LATERAL EDGE RESTRAINT†

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Abstract—An analysis is presented of the initial buckling of rectangular plates which are orthotropic in plane and in bending; the loading is either applied in two directions parallel to the plate edges, or is uniaxial with a degree of restraint applied laterally in the plane of the plate. For long restrained plates it is shown that buckling under load in one direction may actually be caused by the transverse load which is induced in the other direction by restraining the plate Poisson's ratio effect. This may occur for values of the Poisson's ratio conventionally associated with isotropic materials, only partial lateral restraint needing to be present.

NOTATION

a, b	length and width of plate
$A = (A_{ij})$, $D = (D_{ij})$	matrices of in plane and bending stiffnesses
D	bending stiffness of an isotropic plate
E_{11}, E_{22}, G	longitudinal, transverse and shear moduli of a ply of composite material
k	load parameter when N_y acts as a constant preload—see equation (12)
l	load parameter when loads vary proportionally—see equation (14)
m, n	number of half-waves of w in x and y directions
M_x, M_y, M_{xy}	bending moments
N_x, N_y, N_{xy}	middle-surface forces per unit width
$\bar{N}_x, \bar{N}_y$	axial and transverse buckling loads of a rectangular plate—see equations (7), (10)
N'_x, N'_y	axial and transverse buckling loads of an infinite plate—see equations (8), (11)
N''_x	buckling load defined by equation (9)
v, w	displacements in y, z directions
x, y, z	cartesian coordinates
α	$= a/b$ is the plate aspect ratio
δ	arbitrary small constant
$\epsilon_x, \epsilon_y, \gamma_{xy}$	in-plane strains
λ	$= na/mb$
ν	Poisson's ratio
ν_{12}, ν_{21}	in-plane Poisson's ratios of an orthotropic plate
ψ	parameter governing degree of lateral restraint

1. INTRODUCTION

Recent interest in the use of fibre-reinforced composite materials in load-carrying applications has seen an emphasis on application of the stronger and stiffer reinforcements which are currently available. With the enhanced mechanical properties which have been demonstrated, there has been extensive use of thin sections of such materials laminated into thin sheet form and this, in turn, has led to an interest in the stability of laminated sheets under a variety of loadings. A laminated composite sheet is conventionally produced by the combination of a number of thin plies of unidirectionally reinforced material; the elastic response of such a laminate subjected to an applied load is, in general, highly anisotropic and exhibits a coupling between the in-plane and out-of-plane responses.

This paper considers the initial buckling of rectangular simply-supported plates subjected either to a destabilising biaxial load system (N_x, N_y), or to an axial load N_x with a degree of elastic lateral restraint applied to the sides of the plate. Attention is focussed on plates whose properties, both in plane and in bending, are orthotropic; the analysis thus applies to a wide class of laminated plate, as described above, or to homogeneous orthotropic plates in which the elastic properties do not vary through the plate thickness. It is well known in isotropic plates that when a laterally restrained plate is compressed in the axial direction, a destabilising transverse load is induced by the restraint; the plate instability is thus equivalent to that under biaxial loading where a destabilising transverse load is applied in proportion to the axial load. The present paper shows that for orthotropic plates whose elastic constants fall into certain ranges, the transverse load induced by the lateral restraint may not only destabilise the plate, but that buckling may actually be caused by this effect. It is not necessary for this lateral restraint to be completely rigid, and the in-plane Poisson's ratios at which the phenomenon may occur are of the same magnitude as in conventional isotropic materials.

The buckling of an infinitely long orthotropic plate is first considered. For a range of loads, the axial load N_x does not interact at buckling with the transverse load N_y; it is thus shown, for an elastically restrained plate, that the induced transverse load may cause buckling for a certain range of elastic properties and of lateral restraint. It follows that in a finite plate the induced transverse load may be the primary cause, although not the sole cause, of buckling; for plates of sufficient length, a criterion is suggested which will allow the effect to be assessed for finite plates.

2. BASIC EQUATIONS AND INITIAL BUCKLING

The equations governing the stability under biaxial loading of rectangular orthotropic plates having simply-supported edges have been given by Lekhnitskii[1]; Wittrick[2], in considering the same and related problems, has drawn attention to the correlation which exists between the buckling loads of orthotropic and isotropic plates for various edge conditions, including those of simple support. Shulesko[3] has described a reduction method which gives a number of specific solutions for the stability, under uniaxial and biaxial loading, of orthotropic plates whose sides are subject to a variety of boundary conditions, including those of simple support. The present paper considers the stability of both finite and infinite orthotropic plates. To account for the effects of lateral restraint it is necessary to consider the elastic response in the plane of the plate, in addition to that in bending. Refs. 1 and 2 consider only the stability of unrestrained plates; Ref. 3 covers the case where the plate sides are elastically restrained against rotation but in-plane elastic restraint, of the present type, is not considered.

A plate is considered which is orthotropic in plane and in bending, in which the plate axes of symmetry and the elastic axes of symmetry coincide. In general, when the plate is laminated from a number of plies of unidirectionally reinforced material, the orthotropy in plane and in bending will differ; with a coordinate system $0(x, y, z)$ with stress resultants (N_x, N_y, N_{xy}), strains $(\epsilon_x, \epsilon_y, \gamma_{xy})$, moments (M_x, M_y, M_{xy}) and transverse displacement w, the stress-strain and moment-curvature relationships are, respectively,

$$\left.\begin{aligned}
\begin{pmatrix} N_x \\ N_y \\ N_{xy} \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix}, \\
\begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix} &= -\begin{pmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix} \begin{pmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x \partial y \end{pmatrix},
\end{aligned}\right\} \tag{1}$$

A_{11}, A_{12}, A_{22}, A_{33} being the in-plane and D_{11}, D_{12}, D_{22}, D_{33} the bending stiffnesses of the plate. It is assumed that no coupling exists between in-plane and bending effects, so that a laminated plate will have an appropriate lamination sequence through its thickness[4].

2.1 *Biaxially loaded plates*

The equation governing the transverse deflection of an orthotropic plate subjected to biaxial load (N_x, N_y) is[1, 2]

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12}+2D_{33})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2}. \tag{2}$$

A rectangular plate, $0 \leq x \leq a$, $0 \leq y \leq b$, is considered whose sides are simply-supported; the boundary conditions are thus

$$\left.\begin{aligned} w = \frac{\partial^2 w}{\partial x^2} = 0, \qquad (x = 0, a), \\ w = \frac{\partial^2 w}{\partial y^2} = 0, \qquad (y = 0, b). \end{aligned}\right\} \tag{3}$$

An additional boundary condition, of elastic lateral restraint at the sides $y = 0$, b, will be introduced later and will be implemented by ensuring a suitable inter-relationship between (N_x, N_y).

If a buckling mode

$$w = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{4}$$

is assumed, the boundary conditions (3) are satisfied automatically; substitution in equation (2) then gives, at buckling,

$$-\frac{b^2}{n^2}N_x - \frac{a^2}{m^2}N_y = \pi^2\{D_{22}\lambda^2 + 2(D_{12} + 2D_{33}) + D_{11}\lambda^{-2}\} \tag{5}$$

where

$$\lambda = \frac{na}{mb}. \tag{6}$$

Equation (5) may be used in various forms to determine initial buckling under different loading conditions. Under uniaxial load N_x alone, there is one half-wave across the plate width and

$$N_x = \bar{N}_x = -\frac{\pi^2}{b^2}\{D_{22}\lambda^2 + 2(D_{12} + 2D_{33}) + D_{11}\lambda^{-2}\} \tag{7}$$

with $n = 1$ at buckling. The behaviour of an axially loaded orthotropic plate at buckling is well known, successive segments of the buckling curve, given by (6) and (7), being determined by $m = 1,2,3\ldots$ etc. as illustrated in Fig. 1 (curve corresponding to $l = 0$); the properties of the specific plate chosen for illustration in this paper, and used in preparing Fig. 1, are detailed in Section 3 below. For a very long plate in which $\alpha \to \infty$ ($\alpha = a/b$ is the plate aspect ratio), λ takes

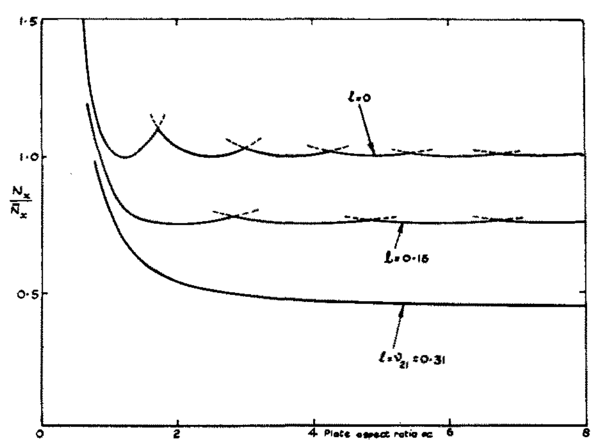

Fig. 1. Variation with plate aspect ratio of N_x at buckling for applied loads which vary proportionally.

the value which minimises the expression on the right of (7), and the critical buckling load is

$$N_x = N_x' = -\frac{2\pi^2}{b^2}\{D_{12} + 2D_{33} + (D_{11}D_{22})^{1/2}\} \tag{8}$$

at a half-wavelength $b(D_{11}/D_{22})^{1/4}$. When $D_{11} = 0$,

$$N_x = N_x'' = -\frac{2\pi^2}{b^2}(D_{12} + 2D_{33}). \tag{9}$$

Under transverse loading N_y alone, there is a single half-wave in the x-direction and equation (5) gives as the critical load

$$N_y = \bar{N}_y = -\frac{\pi^2}{a^2}\{D_{22}\lambda^2 + 2(D_{12} + 2D_{33}) + D_{11}\lambda^{-2}\}, \tag{10}$$

with $m = 1$. For a very long plate ($\alpha \to \infty$, $\lambda \to \infty$), $n = 1$ also and the 'wide-plate' buckling load then follows as

$$N_y = N_y' = -\frac{\pi^2 D_{22}}{b^2}. \tag{11}$$

In considering biaxial loading, there are two ways in which the load may be envisaged to be applied; the cases distinguished [1] are (a) that in which N_y is regarded as a constant preload, only N_x being allowed to vary, and (b) that in which (N_x, N_y) maintain a constant ratio during loading.

If N_y, which by itself is insufficient to destabilise the plate, is considered as a preload defined by

$$N_y = kN_y' \tag{12}$$

then (5) and (11) give, at buckling,

$$N_x = -\frac{\pi^2}{b^2}\{D_{22}(1-k)\lambda^2 + 2(D_{12} + 2D_{33}) + D_{11}\lambda^{-2}\},$$

putting $n = 1$. If λ is chosen to minimise this expression, the axial buckling load of a pre-loaded long plate is derived as

$$N_x = -\frac{2\pi^2}{b^2}\{D_{12} + 2D_{33} + [D_{11}D_{22}(1-k)]^{1/2}\} \tag{13}$$

at a half-wavelength $b\{D_{11}/D_{22}(1-k)\}^{1/4}$.

Alternatively, if biaxial load is applied with constant proportionality maintained between (N_x, N_y), so that

$$N_y = lN_x, \tag{14}$$

equation (5) gives, at buckling,

$$N_x = -\frac{n^2\pi^2}{b^2}(1 + l\lambda^2)^{-1}\{D_{22}\lambda^2 + 2(D_{12} + 2D_{33}) + D_{11}\lambda^{-2}\}; \tag{15}$$

the functional dependence of this expression on the single parameter λ (see equations (5), (7) and (10)) should be noted. It may be shown that the right-hand side of (15) has a minimum for a given λ only when

$$l \leq \frac{D_{22}}{2(D_{12} + 2D_{33})}. \tag{16}$$

For values of l satisfying (16), the buckling behaviour is qualitatively similar to that when $l = 0$. Fig. 1 illustrates this (curve labelled $l = 0{\cdot}15$) for the particular plate detailed below; N_x may be regarded as the applied load, and successive segments of the curve relating N_x to α again correspond to $m = 1,2,3,\ldots$ etc. For very long plates ($\alpha \to \infty$, $n = 1$) the minimum value of (15) gives the corresponding buckling load to be

$$N_x = N_x''' = -\frac{2\pi^2}{b^2}\left(D_{12} + 2D_{33} - lD_{11} + [D_{11}\{D_{22} - 2l(D_{12} + 2D_{33}) + l^2 D_{11}\}]^{1/2}\right) \tag{17}$$

at a half-wavelength

$$b\{D_{22} - 2l(D_{12} + 2D_{33})\}^{-1/2}\{lD_{11} + [D_{11}\{D_{22} - 2l(D_{12} + 2D_{33}) + l^2 D_{11}\}]^{1/2}\}^{1/2}. \tag{18}$$

When l does not satisfy the inequality (16) the variation of buckling load is similar to that illustrated in Fig. 1 (curve $l = 0{\cdot}31$) where it is evident that no minimum exists. In this case, the asymptotic value of (15) gives as the long-plate buckling load

$$N_x = -\frac{\pi^2 D_{22}}{lb^2}. \tag{19}$$

2.2 *Laterally restrained plates*

It will next be shown that the foregoing analysis, of plates in which (N_x, N_y) remain proportional during loading, applies also to laterally restrained plates. The in-plane pre-buckling response is given by equation (1); under biaxial load, the strain ϵ_y is thus given by

$$\epsilon_y = \frac{\partial v}{\partial y} = (A_{11}A_{22} - A_{12}^2)^{-1}(A_{11}N_y - A_{12}N_x)$$

which, on integration, gives for the lateral displacement v,

$$(A_{11}A_{22} - A_{12}^2)v = (A_{11}N_y - A_{12}N_x)(y - b/2). \tag{20}$$

This paper considers plates whose sides $y = 0$, b are subject to elastic restraint against lateral displacement. The in-plane boundary conditions, which ensure that the transverse load N_y and the lateral displacement are proportional, are

$$N_y = \left\{\begin{array}{ll} 2v(A_{11}A_{22} - A_{12}^2)\psi/bA_{11}(1-\psi), & (y = 0), \\ -2v(A_{11}A_{22} - A_{12}^2)\psi/bA_{11}(1-\psi), & (y = b). \end{array}\right\} \tag{21}$$

The parameter ψ is chosen to determine the degree of lateral restraint; $\psi = 0$ corresponds to laterally-free edges ($N_y = 0$) and $\psi = 1$ to infinitely rigid restraint ($v = 0$).

Since the plate, prior to buckling, is uniformly loaded it may be shown from (20) and (21) that the condition of elastic lateral restraint is equivalent to requiring that (N_x, N_y) maintain a constant ratio during loading, with

$$l = \psi A_{12}/A_{11} \tag{22}$$

(see equation (14)). If N_x is regarded to be the applied load, the load N_y derived from (14) and (22) may be considered to be the transverse load induced by the lateral restraint *via* the Poisson's ratio effect. Defining the two Poisson's ratios, ν_{12} and ν_{21}, of the plate by

$$A_{12} = \nu_{21}A_{11} = \nu_{12}A_{22} \tag{23}$$

equation (22) gives

$$l = \psi\nu_{21}; \tag{24}$$

for the rigidly restrained plate ($\psi = 1$), the corresponding proportionality constant governing the applied loads is $l = \nu_{21}$. From equation (24), it thus appears that the parameter ψ may be regarded as a proportional reduction, due to elastic effects, of the full Poisson's ratio effect caused by completely rigid lateral restraint. The buckling load of a laterally restrained plate can thus be derived directly from the above analysis for biaxial loading (equations (14)–(19)) merely by choosing the particular proportionality constant, $\psi\nu_{21}$, given by equation (24). The presence of the particular Poisson's ratio ν_{21} in equation (24) should be noted; when lateral restraint is absent, it is the other Poisson's ratio ν_{12} which governs the familiar ratio of lateral and longitudinal strains *via* the relationship

$$\epsilon_y = \nu_{12}\epsilon_x.$$

3. BUCKLING OF BIAXIALLY LOADED AND RESTRAINED PLATES

The analysis of the present paper is general, but will be illustrated by reference to a symmetrically laminated composite plate having eighteen equal-thickness plies of unidirectional material. The elastic properties of a ply are

$$E_{11} = 207\,\mathrm{GN/m^2},\quad E_{22} = 7{\cdot}58\,\mathrm{GN/m^2},\quad G = 5{\cdot}52\,\mathrm{GN/m^2},\quad \nu_{12} = 0{\cdot}3$$

and the plies are arranged at angles

$$(30°, -30°, -30°, 90°, 30°, 30°, -30°, 90°, 90°|\ \text{symmetric about mid-plane})$$

to the x-axis. This lay-up has equal numbers of plies aligned in each of the directions (−30°, 30°, 90°); the in-plane elastic properties of the plate are therefore isotropic, with

$$\begin{cases} A_{11} = A_{22} = A_{12} + 2A_{33}, \\ \nu_{12} = \nu_{21} = 0{\cdot}31. \end{cases}$$

The plate elastic constants are of the form given in equation (1); in particular, no elastic coupling exists between in-plane and out-of-plane effects, or between bending and twisting.

3.1 *General behaviour*

Under applied biaxial loading, the interaction at buckling between N_x and N_y, governed by equation (5), will be qualitatively the same as that of isotropic plates [5]. Depending on the ranges in which N_x and N_y fall, equation (5) will determine an interaction diagram consisting of a series of linear segments corresponding to $(m, n) = \ldots (1,3), (1,2), (1,1), (2,1), \ldots$ etc; Fig. 2 shows this relationship between $N_x/\bar{N}_x$ and $N_y/\bar{N}_y$ for a plate with $\alpha = 6$ and the elastic properties given above.

In general, the (1, 1) segment (corresponding to AB in Fig. 2) will be governed by

$$N_y + \alpha^{-2}N_x = -\frac{\pi^2}{b^2}\{D_{22} + 2(D_{12} + 2D_{33})\alpha^{-2} + D_{11}\alpha^{-4}\},$$

from equations (5) and (6); this segment will intersect the (2,1) segment (the point B in Fig. 2) where

$$N_x = -\frac{\pi^2}{b^2}\{2(D_{12} + 2D_{33}) + 5D_{11}\alpha^{-2}\},$$

and will intersect the (1, 2) segment where

$$N_x = \frac{\pi^2}{b^2}\{4D_{22}\alpha^2 - D_{11}\alpha^{-2}\}.$$

By considering the limiting form of these three expressions when $\alpha \to \infty$ some aspects of the (N_x, N_y) interaction for a very long plate can be deduced. The (1, 1) segment has equation

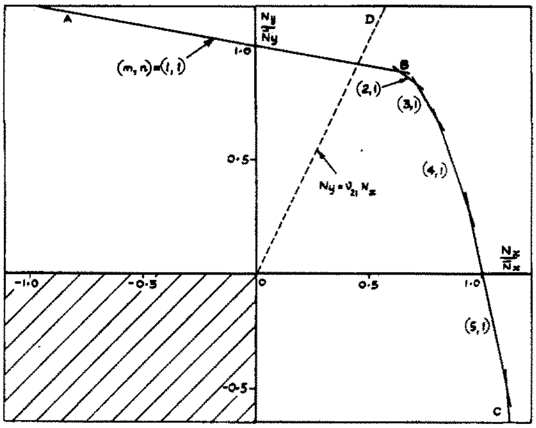

Fig. 2. Interaction at buckling between N_x and N_y for a plate of aspect ratio 6.

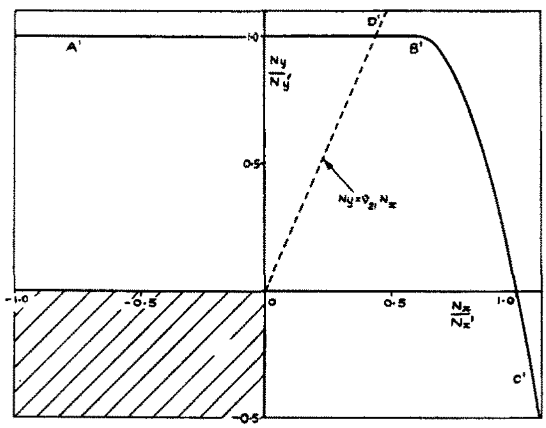

Fig. 3. Interaction at buckling between N_x and N_y for an infinitely long plate.

$N_y = N'_y$ (see equation (11)); it intersects the (2, 1) segment where $N_x = N''_x$ (see equation (9)) whereas the intersection with the (1, 2) segment corresponds to $N_x \to \infty$. The diagram relating N_x/N'_x and $k = N_y/N'_y$ for such a plate, again exemplified by the plate with elastic properties given above, is shown in Fig. 3; the parabolic segment $B'C'$ of the curve is represented by the right-hand side of equation (13) when

$$N_x \geq N''_x, \tag{25}$$

equality corresponding to the point B' (Fig. 3). The linear segment $A'B'$ corresponds to $N_y = N'_y$; for values of N_x, corresponding to points on $A'B'$, which do not satisfy the inequality (25) the values of (N_x, N_y) at buckling do not interact, instability being caused by the action of N_y acting alone. This is also the case for isotropic plates, and the existence of a non-interacting region of (N_x, N_y) is therefore to be expected. The mode shape at buckling due to N_y alone is proportional to $\sin(\pi y/b)$, and this does not vary in the x-direction; thus, for example, tensile values of N_x will not contribute in the $N_x(\partial^2 w/\partial x^2)$ term on the right of equation (2), and the onset of instability will not depend on N_x.

Fig. 3 also shows, for the particular plate considered, the linear segment OD', governed by $N_y = \nu_{21}N_x$, corresponding to completely rigid ($\psi = 1$) lateral restraint of the plate; since this intersects $A'B'$ rather than $B'C'$, the corresponding rigidly-restrained plate buckling under axial load N_x is caused instead by the induced transverse load N_y, with a buckling mode which does not vary along the plate length.

It is not necessary to have completely rigid lateral restraint for this to occur. In general, if the slope of the line corresponding to OD' (Fig. 3) exceeds that of OB', i.e. if

$$\psi\nu_{21} \geq N'_y/N''_x, \tag{26}$$

then buckling under axial load will be due solely to the induced transverse load; for the particular example chosen, (26) corresponds to $\psi \geq 0{\cdot}73$. The inequality (26) will in general be satisfied if either of ν_{21}, the particular load N_x'' or the degree of lateral restraint ψ is sufficiently large, or if the transverse buckling load N_y' is sufficiently small. Expressed in terms of the elastic constants, (26) is equivalent to

$$\psi\nu_{21} \geq \frac{D_{22}}{2(D_{12}+2D_{33})}, \tag{27}$$

from (9) and (11). Alternatively, for the particular case of a homogeneous plate in which the elastic properties do not vary across the plate thickness, (A_{ij}) and (D_{ij}) are proportional and (27) gives

$$2\psi A_{12}(A_{12}+2A_{33}) - A_{11}A_{22} \geq 0. \tag{28}$$

3.2 *Buckling of finite plates*

By first considering a very long plate, in which there is no interaction between (N_x, N_y) over a given range, it has been possible to show that the induced transverse load is the cause of instability in a restrained plate. For finite plates, the inequality (27) may be regarded as the necessary condition for the possible occurrence of such a 'Poisson's-ratio' instability, provided the plate is sufficiently long; the variation of axial buckling load with α for the particular plate above having rigid lateral restraint ($\psi = 1$) is illustrated in Fig. 1 (curve corresponding to $l = \nu_{21} = 0{\cdot}31$).

Fig. 3 represents the limiting form when $\alpha \to \infty$ of an interaction diagram such as Fig. 2, the parabolic arc $B'C'$ being the limiting form of a sequence of segments, such as BC in Fig. 2, with $n = 1$ and $m = 2, 3, \ldots$ etc. Thus it is to be expected that buckling caused primarily by the Poisson's ratio effect will be possible in finite plates, and that this instability will be associated with an $(m, n) = (1, 1)$ buckling mode represented by a linear segment such as AB (Fig. 2). However, N_x and N_y will now couple, and buckling will be caused also partly by the action of N_x; a suitable criterion for buckling to remain 'due' to the lateral restraint is if, starting at a restrained buckling load N_x,

(i) complete removal of the axial load N_x, together with

(ii) associated increase of the induced transverse load by an amount δ to become $(1+\delta)N_y$, δ being a suitably chosen arbitrary small quantity,

causes buckling.

This is next considered. When buckling of a restrained plate occurs under axial load N_x, the induced transverse load is

$$N_y = -\frac{\pi^2}{b^2}\psi\nu_{21}(1+\psi\nu_{21}\alpha^2)^{-1}\{D_{22}\alpha^2 + 2(D_{12}+2D_{33}) + D_{11}\alpha^{-2}\}, \tag{29}$$

from (14), (15) and (24) with $m = n = 1$; the transverse load which, acting alone, would cause buckling is

$$\bar{N}_y = -\frac{\pi^2}{b^2}\alpha^{-2}\{D_{22}\alpha^2 + 2(D_{12}+2D_{33}) + D_{11}\alpha^{-2}\}, \tag{30}$$

from (10). On the basis of the criterion stated above it follows that buckling may be regarded as 'caused' by the lateral restraint when $(1+\delta)N_y$ (N_y being given by (29)) exceeds $\bar{N}_y$; this occurs if

$$\alpha \geq (\psi\nu_{21}\delta)^{-1/2}. \tag{31}$$

Apart from the parameter δ, (31) depends only on ψ and ν_{21}; it may thus be seen that, however small the chosen value of δ, (31) will always hold for plates which are sufficiently long.

For the elastic properties of the example chosen, the inequality (31) with $\psi = 1$ and $\delta = 0{\cdot}1$

gives $\alpha \geq 5{\cdot}7$. Poisson's-ratio buckling may thus be regarded as occurring in the plate ($\alpha = 6$) illustrated in Fig. 2 where it will be seen that the line OD, corresponding to $\psi = 1$ and $N_y = \nu_{21} N_x$, intersects the segment AB corresponding to $(m, n) = (1, 1)$.

Occurrence of the restrained-plate buckling phenomenon described is not dependent on unusually large values of the Poisson's ratio, such as those known to occur in laminated plates for some types of layup; in the specific example chosen, the in-plane isotropy ensures a Poisson's ratio ($\nu_{12} = \nu_{21} = 0{\cdot}31$) typical of isotropic materials, the phenomenon occurring due to the marked difference between the axial and transverse buckling loads ($N_x'' = 4{\cdot}43\ N_y'$, see the inequality (26)).

The foregoing analysis is developed on the assumption of loading in the x-direction and restraint in the y-direction. For a plate loaded in the y-direction and restrained in the x-direction a modified analysis could be derived in an obvious manner; alternatively, the analysis above applies with the elastic properties 'rotated' through 90°, i.e. with $(D_{11}, D_{22}, A_{11}, A_{22}, \nu_{12}, \nu_{21})$ replaced by $(D_{22}, D_{11}, A_{22}, A_{11}, \nu_{21}, \nu_{12})$ respectively.

4. HOMOGENEOUS ISOTROPIC PLATES

The analysis of this paper applies equally to homogeneous isotropic plates for which it is merely necessary in the above analysis to make (A_{ij}) and (D_{ij}) proportional, with

$$\left.\begin{aligned} D_{11} = D_{22} = D_{12} + 2D_{33} = D, \\ \nu_{12} = \nu_{21} = \nu. \end{aligned}\right\} \qquad (32)$$

From (27) and (32), buckling can only be caused in such a plate under rigid lateral restraint if $\nu \geq \frac{1}{2}$, which implies unrealistic values of the elastic constants; partial lateral restraint ($\psi < 1$) implies even higher ν values. The axial buckling load of a restrained rectangular plate is given by

$$N_x = -\frac{\pi^2 D}{b^2}(1 + \nu\psi m^{-2}\alpha^2)^{-1}\left(\frac{\alpha}{m} + \frac{m}{\alpha}\right)^2, \qquad (33)$$

from (15) and (22); for the rigidly-restrained plate ($\psi = 1$) this agrees with the result given by Przemieniecki[6].

5. CONCLUSIONS

The conclusions of this paper may be summarised as follows:

(i) Buckling of infinitely long laterally restrained orthotropic plates, of the type considered, may be caused solely by the transverse load induced by the Poisson's ratio effect if the plate elastic properties satisfy the inequality (27). (ii) In a finite plate, which is long enough for the inequality (31) to be satisfied in addition to (27), buckling may be regarded as caused primarily by this induced transverse load. (iii) Unusually large Poisson's ratio values are not necessary for this phenomenon to occur; it can also follow from a disparity between the axial and transverse buckling loads of the unrestrained plate. (iv) The phenomenon may occur for partial elastic restraint, as well as for plates which are rigidly restrained laterally.

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